

On free infinite divisibility for classical Meixner distributions

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Abstract

We prove that symmetric Meixner distributions, whose probability densities are proportional to $|\Gamma(t+ix)|^2$, are freely infinitely divisible for $0 < t \leq \frac{1}{2}$. The case $t = \frac{1}{2}$ corresponds to the law of Lévy's stochastic area whose probability density is $\frac{1}{\cosh(\pi x)}$. A logistic distribution, whose probability density is proportional to $\frac{1}{\cosh^2(\pi x)}$, is freely infinitely divisible too.

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1 Introduction

The free convolution $\mu \boxplus \nu$ of probability measures μ and ν on \mathbb{R} is the distribution of $X + Y$, where X and Y are free self-adjoint random variables respectively following the distributions μ and ν . A probability measure ν on \mathbb{R} is said to be *freely infinitely divisible* if, for any $n \in \{1, 2, 3, \dots\}$, there exists ν_n such that

$$\nu = \underbrace{\nu_n \boxplus \dots \boxplus \nu_n}_{n \text{ times}}.$$

This concept was introduced in [V86] and its basic characterization was established in [BV93]. The most important freely infinitely distributions are Wigner's semicircle law and the free Poisson law.

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Recent work has increased examples of probability measures which are infinitely divisible in both senses, classical and free: the Gaussian distribution [BBL11], chi-square distribution $\frac{1}{\sqrt{\pi x}} e^{-x} 1_{(0,\infty)}(x) dx$ [AHS], positive Boolean stable law with stability index $\alpha \in (0, \frac{1}{2}]$ [AHb] and Student distribution $\frac{1}{B(\frac{1}{2}, n-\frac{1}{2})} \frac{1}{(1+x^2)^n} 1_{\mathbb{R}}(x) dx$ for $n = 1, 2, 3, \dots$ [H]. A still remaining question is whether there is a general theory on the intersection of free and classical infinite divisibility. We will add two more examples, Meixner distributions and the logistic distribution, which may contribute to a solution.

We will prove that symmetric *Meixner distributions*

$$\rho_t(dx) := \frac{4^t}{2\pi\Gamma(2t)} |\Gamma(t+ix)|^2 dx, \quad x \in \mathbb{R}$$

are freely infinitely divisible for $0 < t \leq \frac{1}{2}$. The measures ρ_t are probability distributions of a Lévy process, called a Meixner process [ST98], since the characteristic function of ρ_t is given by

$$\widehat{\rho}_t(z) = \left(\frac{1}{\cosh(\frac{z}{2})} \right)^{2t}. \quad (1.1)$$

Hence ρ_t is classically infinitely divisible for any $t > 0$. The measure ρ_t orthogonalizes Meixner-Pollaczek polynomials $\{P_n^{(t)}(x)\}_{n=0}^\infty$ which satisfy the recurrence relation [KLS10]

$$xP_n^{(t)}(x) = P_{n+1}^{(t)}(x) + \frac{n(n+2t-1)}{4} P_{n-1}^{(t)}(x), \quad n \geq 1,$$

with initial conditions $P_0^{(t)}(x) = 1$, $P_1^{(t)}(x) = x$.

If $t = \frac{1}{2}$, the measure $\rho_{1/2}$ coincides with

$$\mu_1(dx) = \frac{1}{\cosh(\pi x)} dx, \quad x \in \mathbb{R},$$

which is the law of *Lévy's stochastic area*¹

$$\frac{1}{2} \int_0^1 (B_t^1 dB_t^2 - B_t^2 dB_t^1),$$

where (B_t^1, B_t^2) is a standard two-dimensional Brownian motion [L51]. The moments m_n of the rescaled measure $\frac{1}{2 \cosh(\pi x/2)} dx$ are *Euler numbers* (with positive signs):

$$(m_0, m_2, m_4, m_6, m_8, \dots) = (1, 1, 5, 61, 1385, 50521, \dots), \quad m_{2n+1} = 0, \quad n \geq 0.$$

See [AS70, Chapter 23] for Euler numbers.

The *logistic distribution*

$$\mu_2(dx) = \frac{\pi}{2 \cosh^2(\pi x)} dx, \quad x \in \mathbb{R},$$

¹This measure is also called the *hyperbolic secant distribution*.

is known to be classically infinitely divisible [B92], and we are going to prove that it is freely infinitely divisible too. This measure orthogonalizes *continuous Hahn polynomials* $\{P_n(x)\}_{n=0}^\infty$ which satisfy the recurrence relation [KLS10]

$$xP_n(x) = P_{n+1}(x) + \frac{n^4}{4(4n^2 - 1)}P_{n-1}(x), \quad n \geq 1,$$

with initial conditions $P_0(x) = 1$, $P_1(x) = x$.

The moments m'_n of the rescaled measure $\frac{\pi}{4 \cosh^2(\pi x/2)} dx$ are

$$(m'_0, m'_2, m'_4, m'_6, m'_8, \dots) = \left(1, \frac{1}{3}, \frac{7}{15}, \frac{31}{21}, \frac{127}{15}, \dots\right), \quad m'_{2n+1} = 0, \quad n \geq 0,$$

which can be written as $m'_n = |(2 - 2^n)B_n|$ in terms of *Bernoulli numbers* B_n [AS70].

2 Preliminaries

Let \mathbb{C}^+ and \mathbb{C}^- be the upper half-plane and the lower half-plane respectively. Basic tools for proving free infinite divisibility are the Cauchy transform

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - x} \mu(dx), \quad z \in \mathbb{C}^+$$

and its reciprocal $F_\mu(z) := \frac{1}{G_\mu(z)}$. The reciprocal Cauchy transform maps \mathbb{C}^+ to \mathbb{C}^+ analytically, and it satisfies $\text{Im } F_\mu(z) \geq \text{Im } z$ for $z \in \mathbb{C}^+$. For any $\beta > 0$, there exists $M > 0$ such that F_μ has a right inverse map F_μ^{-1} in $\Gamma_{\beta, M} := \{z \in \mathbb{C}^+ : \text{Im } z > M, \beta |\text{Re } z| < \text{Im } z\}$ such that $F_\mu^{-1}(\Gamma_{\beta, M}) \subset \mathbb{C}^+$ and $F_\mu \circ F_\mu^{-1} = \text{Id}$ [BV93]. In terms of analytic properties of F_μ^{-1} , a useful subclass of freely infinitely divisible distributions is introduced.

Definition 2.1. A probability measure μ is said to be in the class \mathcal{UI} if F_μ^{-1} defined in a domain of the form $\Gamma_{\beta, M}$ analytically extends to a univalent map in \mathbb{C}^+ . Equivalently, $\mu \in \mathcal{UI}$ if and only if there exists a simply connected open set $\mathbb{C}^+ \subset \Omega \subset \mathbb{C}$ such that

- (i) F_μ analytically extends to a univalent map in Ω ,
- (ii) $F_\mu(\Omega) \supset \mathbb{C}^+$.

This equivalence is proved just by applying Riemann mapping theorem.

Remark 2.2. In [AHa] we required F_μ to be univalent in \mathbb{C}^+ in the definition of $\mu \in \mathcal{UI}$, but this automatically follows. If F_μ^{-1} is analytic in \mathbb{C}^+ , then $F_\mu^{-1} \circ F_\mu(z) = z$ for $z \in \mathbb{C}^+$ by Identity Theorem, so that F_μ is univalent in \mathbb{C}^+ .

Lemma 2.3 ([AHa]). (1) *If $\mu \in \mathcal{UI}$, then μ is freely infinitely divisible.*

(2) *The class \mathcal{UI} is closed with respect to the weak convergence.*

(3) *The class \mathcal{UI} is not closed under free convolution, i.e. $\mu, \nu \in \mathcal{UI}$ does not imply $\mu \boxplus \nu \in \mathcal{UI}$.*

This class was essentially introduced in [BBL11] to show that the normal law is freely infinitely divisible, and this class has been successfully applied to several probability measures [ABBL10, AB, AHa, AHb, H]. Examples are presented below, mostly taken from the aforementioned references.

Example 2.4. The following probability measures belong to \mathcal{UI} .

(1) Wigner's semicircle law

$$\mathbf{w}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]}(x) dx, \quad F_{\mathbf{w}}^{-1}(z) = z + \frac{1}{z}.$$

(2) The free Poisson law (or Marchenko-Pastur law)

$$\mathbf{m}(dx) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{[0,4]}(x) dx, \quad F_{\mathbf{m}}^{-1}(z) = z + \frac{z}{z-1}.$$

(3) The Cauchy distribution

$$\mathbf{c}(dx) = \frac{1}{\pi(1+x^2)} 1_{\mathbb{R}}(x) dx, \quad F_{\mathbf{c}}^{-1}(z) = z - i.$$

(4) [AHa] The beta distribution

$$\beta_a(dx) = \frac{\sin(\pi a)}{\pi a} \left(\frac{1-x}{x} \right)^a 1_{[0,1]}(x) dx, \quad F_{\beta_a}^{-1}(z) = \frac{1}{1 - (1 - \frac{a}{z})^{\frac{1}{a}}}$$

for $\frac{1}{2} \leq |a| < 1$. $\beta_{\frac{1}{2}}$ is equal to \mathbf{m} up to scaling.

(5) [BBL11] The Gaussian distribution

$$\mathbf{g}(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} 1_{\mathbb{R}}(x) dx.$$

(6) [ABBL10] The q -Gaussian distribution

$$\mathbf{g}_q(dx) = \frac{\sqrt{1-q}}{\pi} \sin \theta(x) \prod_{n=1}^{\infty} (1-q^n) |1 - q^n e^{2i\theta(x)}|^2 1_{[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]}(x) dx$$

for $q \in [0, 1)$, where $\theta(x)$ is the solution of $x = \frac{2}{\sqrt{1-q}} \cos \theta$, $\theta \in [0, \pi]$. When $q \rightarrow 1$, \mathbf{g}_q converges weakly to \mathbf{g} , and \mathbf{g}_0 coincides with \mathbf{w} . For $q \in (0, 1)$, the density function of \mathbf{g}_q can be written as [LM95]

$$\frac{1}{2\pi} q^{-\frac{1}{8}} (1-q)^{\frac{1}{2}} \Theta_1 \left(\frac{\theta(x)}{\pi}, \frac{1}{2\pi i} \log q \right),$$

where $\Theta_1(z, \tau) := 2 \sum_{n=0}^{\infty} (-1)^n (e^{i\pi\tau})^{(n+\frac{1}{2})^2} \sin(2n+1)\pi z$ is a Jacobi theta function.

(7) [AB] The ultraspherical distribution

$$\mathbf{u}_n(dx) = \frac{1}{16^n B(n + \frac{1}{2}, n + \frac{1}{2})} (4 - x^2)^{n-\frac{1}{2}} 1_{[-2,2]}(x) dx, \quad n = 1, 2, 3, 4, \dots,$$

where $B(p, q)$ is the beta function. The semicircle law \mathbf{w} appears in the case $n = 1$ and the normal law \mathbf{g} in the limit $n \rightarrow \infty$ if \mathbf{u}_n are suitably scaled.

(8) [H] The Student distribution

$$\mathbf{t}_n(dx) = \frac{1}{B(\frac{1}{2}, n - \frac{1}{2})} \frac{1}{(1 + x^2)^n} 1_{\mathbb{R}}(x) dx, \quad n = 1, 2, 3, \dots$$

\mathbf{t}_1 coincides with \mathbf{c} , and if suitably scaled, \mathbf{t}_n weakly converge to \mathbf{g} as $n \rightarrow \infty$.

(9) [AHb] The Boolean stable law

$$\frac{d\mathbf{b}_\alpha^\rho}{dx} = \begin{cases} \frac{\sin(\pi\rho\alpha)}{\pi} \frac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha \cos(\pi\rho\alpha) + 1}, & x > 0, \\ \frac{\sin(\pi(1-\rho)\alpha)}{\pi} \frac{|x|^{\alpha-1}}{|x|^{2\alpha} + 2|x|^\alpha \cos(\pi(1-\rho)\alpha) + 1}, & x < 0, \end{cases}$$

for $0 < \alpha \leq \frac{1}{2}$, $\rho \in [0, 1]$.

If $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$ and $2 - \frac{1}{\alpha} \leq \rho \leq \frac{1}{\alpha} - 1$, the Boolean stable law \mathbf{b}_α^ρ (defined as above too) is still freely infinitely divisible, but not in the class \mathcal{UI} [AHb]. However, most of the known freely infinitely divisible distributions belong to \mathcal{UI} as presented above.

In order to prove $\mu \in \mathcal{UI}$, the following sufficient condition is useful.

Proposition 2.5. *A probability measure μ on \mathbb{R} is in \mathcal{UI} if there exists an infinite curve $\gamma \subset \overline{\mathbb{C}^-}$ with the following properties:*

- (A) γ does not have self-intersection, both edges of γ go to infinity, and there is a sequence $\{R_n\}_{n \geq 1}$ going to infinity as $n \rightarrow \infty$ such that γ and each circle $\{z \in \mathbb{C} : |z| = R_n\}$ intersect at just two points;
- (B) F_μ is injective on γ ;
- (C) $F_\mu(\gamma) \subset \overline{\mathbb{C}^-}$;
- (D) F_μ extends to an analytic function in $D(\gamma)$ and to a continuous function on $\overline{D(\gamma)}$, where $D(\gamma)$ denotes the simply connected open set containing \mathbb{C}^+ with boundary γ ;
- (E) $F_\mu(z) = z + o(z)$ uniformly as $z \rightarrow \infty$, $z \in D(\gamma)$.

Proof. A key is the following fact, which says if an analytic function is injective on a simple closed curve, then it is injective also inside the domain surrounded by the curve.

Lemma 2.6. *Let $C \subset \mathbb{C}$ be a Jordan closed curve and $d(C)$ be the bounded open set surrounded by C . Let $f : d(C) \rightarrow \mathbb{C}$ be an analytic map which extends to a continuous map from $\overline{d(C)}$ to \mathbb{C} . If f is injective on C , then f is injective on $\overline{d(C)}$ and $f(d(C))$ is a bounded Jordan domain surrounded by the Jordan curve $f(C)$.*

The reader is referred to [B79], pp. 310, where the above fact is proved for $C = \{z \in \mathbb{C} : |z| = 1\}$. The general case follows from the Carathéodory theorem; see [B79], pp. 309. The bounded set $D(\gamma)_n := \{z \in D(\gamma) : |z| < R_n\}$ is a Jordan domain thanks to (A). From (B) and (E), one can show that F_μ is injective on the boundary of $D(\gamma)_n$, so that F_μ is univalent in $\overline{D(\gamma)_n}$ from Lemma 2.6 and hence in $\overline{D(\gamma)}$ by taking the limit $n \rightarrow \infty$. Condition (C) implies that $F_\mu(D(\gamma)) \supset \mathbb{C}^+$. So we can take $\Omega = D(\gamma)$, to conclude that $\mu \in \mathcal{UI}$. \square

3 Proof for Meixner distributions

We present some properties of Meixner distributions.

(1) ρ_t is a probability measure for $t > 0$ because

$$\begin{aligned} \int_{\mathbb{R}} |\Gamma(t + ix)|^2 dx &= \int_{\mathbb{R}} \left| \int_0^\infty s^{t+ix-1} e^{-s} ds \right|^2 dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{tu-e^u} e^{ixu} du \right|^2 dx \\ &= 2\pi \int_{\mathbb{R}} e^{2tu-2e^u} du = 2\pi \int_0^\infty \left(\frac{s}{2}\right)^{2t} e^{-s} \frac{ds}{s} = \frac{2\pi\Gamma(2t)}{4^t}, \end{aligned}$$

where Plancherel's theorem was used.

(2) $\rho_{1/2}$ coincides with μ_1 thanks to the formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$.

(3) By residue theorem, $G_t := G_{\rho_t}$ has the series expansion

$$G_t(z) = \frac{4^t}{\Gamma(2t)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+2t)}{n!} \cdot \frac{1}{z + i(t+n)},$$

which is convergent for $0 < t \leq 1/2$.

(4) For any compact set $I \subset \mathbb{R}$, there is $M > 0$ such that

$$|\Gamma(t + zi)\Gamma(t - zi)| \leq M e^{-\pi|x|} |x|^{2t-1}, \quad z = x + yi, \quad |x| \geq 1, \quad t, y \in I.$$

This estimate follows from Stirling's formula.

(5) The density function of ρ_t is symmetric, and moreover strictly decreasing on $[0, \infty)$ as the following calculation shows. We have $\frac{d}{dx} |\Gamma(t + xi)|^2 = -2|\Gamma(t + xi)|^2 \operatorname{Im} \psi(t + xi)$ by using the digamma function $\psi(z) = \frac{d}{dz} \log \Gamma(z)$. It is known that $\psi(z) = -\gamma - \sum_{n=0}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1} \right)$, where γ is Euler's constant, and so $\operatorname{Im} \psi(t + xi) = \sum_{n=0}^{\infty} \frac{x}{(t+n)^2 + x^2} > 0$ for $x > 0$.

We do not use the series expansion of $G_t(z)$; instead the following recursive relation is useful.

Proposition 3.1. *It holds that $G_t(z - ti) = \frac{1}{z} + \frac{it}{z} G_{t+\frac{1}{2}}(z + (\frac{1}{2} - t)i)$ for $t > 0$. Iterative use of this relation extends G_t to a meromorphic function with poles at $-(t+n)i$, $n = 0, 1, 2, \dots$.*

Proof. Assume $t > \frac{1}{2}$. Because $\Gamma(t + iz)\Gamma(t - iz)$ does not have a pole in $\{z \in \mathbb{C} : -\frac{1}{2} \leq \operatorname{Im} z \leq 0\}$ and vanishes rapidly as $\operatorname{Re} z \rightarrow \infty$ (see the above property (4)),

$$\begin{aligned} G_t\left(z - \frac{i}{2}\right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z - (x + \frac{i}{2})} \Gamma(t + ix)\Gamma(t - ix) dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z - x} \Gamma\left(t + \frac{1}{2} + ix\right) \Gamma\left(t - \frac{1}{2} - ix\right) dx, \quad \operatorname{Im} z > \frac{1}{2}. \end{aligned}$$

By using the basic relation $z\Gamma(z) = \Gamma(z + 1)$, we obtain

$$\begin{aligned} G_t\left(z - \frac{i}{2}\right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{\Gamma\left(t + \frac{1}{2} + ix\right) \Gamma\left(t + \frac{1}{2} - ix\right)}{(z - x)(t - \frac{1}{2} - ix)} dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z + (t - \frac{1}{2})i} \left(\frac{1}{t - \frac{1}{2} - ix} - \frac{1}{iz - ix} \right) \left| \Gamma\left(t + \frac{1}{2} + ix\right) \right|^2 dx \\ &= \frac{ti}{z + (t - \frac{1}{2})i} \cdot \frac{4^{t+\frac{1}{2}}}{2\pi\Gamma(2t+1)} \int_{\mathbb{R}} \frac{1}{z - x} \left| \Gamma\left(t + \frac{1}{2} + ix\right) \right|^2 dx \\ &\quad + \frac{1}{(z + (t - \frac{1}{2})i)} \cdot \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma\left(t + \frac{1}{2} + ix\right)|^2}{t - \frac{1}{2} - ix} dx. \end{aligned}$$

In the last integral, we can again apply the formula $z\Gamma(z) = \Gamma(z + 1)$, and moreover we deform the contour \mathbb{R} to $\mathbb{R} + \frac{i}{2}$:

$$\begin{aligned} \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma\left(t + \frac{1}{2} + ix\right)|^2}{t - \frac{1}{2} - ix} dx &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma\left(t + \frac{1}{2} + ix\right) \Gamma\left(t - \frac{1}{2} - ix\right) dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma(t + ix) \Gamma(t - ix) dx \\ &= 1. \end{aligned}$$

The above calculations amount to $G_t\left(z - \frac{i}{2}\right) = \frac{1}{z + (t - \frac{1}{2})i} + \frac{it}{z + (t - \frac{1}{2})i} G_{t+\frac{1}{2}}(z)$ and so to the desired relation, which holds for any $t > 0$ since $G_t(z)$ depends on $t > 0$ real analytically. \square

Lemma 3.2. *If a probability measure μ has a density $p(x)$ such that $p(x) = p(-x)$, $p'(x) \leq 0$ for a.e. $x > 0$ and $\lim_{x \rightarrow \infty} p(x) \log x = 0$, then it holds that $\operatorname{Re} G_{\mu}(x + yi) > 0$, $\frac{\partial}{\partial x} \operatorname{Im} G_{\mu}(x + yi) > 0$ for $x, y > 0$.*

Proof. The proof is just by computation:

$$\begin{aligned} \operatorname{Re} G_{\mu}(x + yi) &= \int_{\mathbb{R}} \frac{x - u}{(x - u)^2 + y^2} p(u) du = -\frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial}{\partial u} \log((x - u)^2 + y^2) \right) p(u) du \\ &= \frac{1}{2} \int_{\mathbb{R}} \log((x - u)^2 + y^2) p'(u) du \\ &= \frac{1}{2} \int_0^{\infty} \log\left(\frac{(x + u)^2 + y^2}{(x - u)^2 + y^2}\right) (-p'(u)) du > 0, \quad x, y > 0. \end{aligned}$$

The property $p'(-u) = -p'(u)$ was used at the final equality. Similarly,

$$\begin{aligned}
\frac{\partial}{\partial x} \operatorname{Im} G_\mu(x + yi) &= \int_{\mathbb{R}} \frac{\partial}{\partial x} \frac{-y}{(x-u)^2 + y^2} p(u) du = \int_{\mathbb{R}} \frac{\partial}{\partial u} \frac{y}{(x-u)^2 + y^2} p(u) du \\
&= - \int_{\mathbb{R}} \frac{y}{(x-u)^2 + y^2} p'(u) du \\
&= - \int_0^\infty \left(\frac{y}{(x-u)^2 + y^2} - \frac{y}{(x+u)^2 + y^2} \right) p'(u) du \\
&= \int_0^\infty \frac{4xyu}{((x-u)^2 + y^2)((x+u)^2 + y^2)} (-p'(u)) du > 0, \quad x, y > 0.
\end{aligned}$$

□

Theorem 3.3. *The Meixner distribution ρ_t is in \mathcal{UI} for $0 < t \leq \frac{1}{2}$.*

Proof. We may assume that $0 < t < \frac{1}{2}$ since the set \mathcal{UI} is closed with respect to the weak convergence. We will check conditions (A)–(E) for $F_t(z) := \frac{1}{G_t(z)}$ and $\gamma_t := \{x - ti : x \in \mathbb{R}\}$. (A) is trivial. To prove (B) and (C), let $f(x) := \operatorname{Re} G_t(x - ti)$, $g(x) := \operatorname{Im} G_t(x - ti)$, $\tilde{f}(x) := \operatorname{Re} G_{t+\frac{1}{2}}(x + (\frac{1}{2} - t)i)$ and $\tilde{g}(x) := \operatorname{Im} G_{t+\frac{1}{2}}(x + (\frac{1}{2} - t)i)$. Then

$$f(x) = \frac{1}{x} - \frac{t}{x} \tilde{g}(x), \quad g(x) = \frac{t}{x} \tilde{f}(x).$$

We can prove the following: (a) $f(x) > 0$, $x > 0$; (b) $g(x) > 0$, $x > 0$; (c) $f'(x) < 0$, $x > 0$. Since $\frac{d}{dx} |\Gamma(t + \frac{1}{2} + xi)|^2 < 0$ for $x > 0$, we can use Lemma 3.2 to assert that $\tilde{f}(x) > 0$, $\tilde{g}'(x) > 0$ for $x > 0$. Hence (b) and condition (C) hold. Since $\tilde{g}(x) < 0$, we have (c):

$$f'(x) = -\frac{1}{x^2} + \frac{t}{x^2} \tilde{g}(x) - \frac{t}{x} \tilde{g}'(x) < 0, \quad x > 0.$$

The properties (a) and (c) imply that F_t is injective on γ_t , condition (B).

From Proposition 3.1, G_t is a meromorphic function and so is F_t . If G_t had a zero in $\overline{D(\gamma_t)}$, there would be a point $z_0 \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$ such that $G_t(z_0 - ti) = 0$. This implies $1 + tiG_{t+\frac{1}{2}}(z_0 + (\frac{1}{2} - t)i) = 0$ and so $G_{t+\frac{1}{2}}(z_0 + (\frac{1}{2} - t)i) = \frac{i}{t} \in \mathbb{C}^+$. This is a contradiction because $G_{t+\frac{1}{2}}$ maps \mathbb{C}^+ into \mathbb{C}^- . Thus condition (D) is proved.

Condition (E) can be checked as follows. Let $p_t(x)$ be the density function of ρ_t . In the integral $\int_{\mathbb{R}} \frac{1}{z-x} \rho_t(dx)$, one is allowed to replace the contour \mathbb{R} by $C_t := \{x - \frac{3t}{2}i : -\infty < x < -\frac{3t}{2}\} \cup \{-\frac{3t}{2}i + \frac{3t}{2}e^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{x - \frac{3t}{2}i : \frac{3t}{2} < x < \infty\}$:

$$\int_{\mathbb{R}} \frac{1}{z-x} \rho_t(dx) = \int_{C_t} \frac{1}{z-w} p_t(w) dw.$$

Clearly $1 = \int_{\mathbb{R}} p_t(x) dx = \int_{C_t} p_t(w) dw$, so we have $1 - zG_t(z) = \int_{C_t} \frac{1}{w-z} w p_t(w) dw$. If z tends to ∞ satisfying $z \in D(\gamma_t)$, then $1 - zG_t(z)$ tends to 0 by Lebesgue convergence theorem. This implies $\left| \frac{F_t(z)-z}{z} \right| \rightarrow 0$, the conclusion. □

Remark 3.4. The free cumulant sequence $(r_n(\mu))_{n=1}^\infty$ of a probability measure μ with finite moments of all orders can be defined as the coefficients of series expansion of $F_\mu^{-1}(z) - z$:

$$F_\mu^{-1}(z) - z = \sum_{n=1}^{\infty} \frac{r_n(\mu)}{z^{n-1}},$$

see [NS06, Remark 16.18]. The free infinite divisibility of ρ_t ($0 < t \leq \frac{1}{2}$) implies that the corresponding free cumulant sequence is conditionally nonnegative definite, i.e. the $N \times N$ matrix $(r_{m+n}(\rho_t))_{m,n=1}^N$ is nonnegative definite for any $N \geq 1$; see Theorem 13.16 of [NS06].² If $t = \frac{1}{2}$, the free cumulants up to the 10th order are given by

$$(r_2(\mu_2), r_4(\mu_1), r_6(\mu_1), \dots) = (1, 3, 38, 947, 37394, \dots), \quad r_{2n+1}(\mu_1) = 0, \quad n \geq 0.$$

This sequence can be found in [OEIS].

4 Proof for the logistic distribution

The free infinite divisibility of the logistic distribution μ_2 is proved with direct computation of the Cauchy transform. From residue theorem, it turns out that

$$\begin{aligned} G_{\mu_2}(z) &= \sum_{n=1}^{\infty} \frac{i}{(z + (n - \frac{1}{2})i)^2} \\ &= \sum_{n=1}^{\infty} \frac{2x(y + n - \frac{1}{2})}{[x^2 + (y + n - \frac{1}{2})^2]^2} + i \sum_{n=1}^{\infty} \frac{x^2 - (y + n - \frac{1}{2})^2}{[x^2 + (y + n - \frac{1}{2})^2]^2}, \quad z = x + yi \in \mathbb{C}^+. \end{aligned} \quad (4.1)$$

Now we take $\gamma_{1/2} := \{x - \frac{i}{2} : x \in \mathbb{R}\}$. The real and imaginary parts of G_{μ_2} on $\gamma_{1/2}$ can be written as

$$\begin{aligned} f(x) &:= \operatorname{Re} G_{\mu_2} \left(x - \frac{i}{2} \right) = \sum_{n=1}^{\infty} \frac{2nx}{(x^2 + n^2)^2}, \\ g(x) &:= \operatorname{Im} G_{\mu_2} \left(x - \frac{i}{2} \right) = \sum_{n=0}^{\infty} \frac{x^2 - n^2}{(x^2 + n^2)^2}. \end{aligned}$$

Fortunately, the imaginary part g can be written by elementary functions.

Lemma 4.1. *The function g is given by $g(x) = \frac{1}{2} \left(\frac{1}{x^2} + \left(\frac{\pi}{\sinh(\pi x)} \right)^2 \right)$.*

Proof. It is known that $\frac{1}{\sinh(\pi x)} = \frac{1}{\pi x} - \frac{\pi}{6}x + O(x^3)$ as $x \rightarrow 0$, and so $\left(\frac{\pi}{\sinh(\pi x)} \right)^2 = \frac{1}{x^2} + O(1)$, $x \rightarrow 0$. The poles of $\left(\frac{\pi}{\sinh(\pi x)} \right)^2$ are at $x = ni$ ($n \in \mathbb{Z}$) and the function $\left(\frac{\pi}{\sinh(\pi x)} \right)^2 -$

²If a measure μ has a compact support, the free infinite divisibility is equivalent to the conditional nonnegative definiteness of free cumulants. This equivalence can be extended to a measure with noncompact support when the moment problem is determinantal.

$\sum_{n=-\infty}^{\infty} \frac{1}{(x-ni)^2}$ does not have a singular point. This function is bounded by a constant on \mathbb{C} and so equal to a constant, which is actually zero as is known from the limit $x \rightarrow \infty$. Hence

$$\begin{aligned} \left(\frac{\pi}{\sinh(\pi x)} \right)^2 &= \sum_{n=-\infty}^{\infty} \frac{1}{(x-ni)^2} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \left(\frac{1}{(x-ni)^2} + \frac{1}{(x+ni)^2} \right) \\ &= \frac{1}{x^2} + 2 \sum_{n=1}^{\infty} \frac{x^2 - n^2}{(x^2 + n^2)^2}, \end{aligned}$$

leading to the conclusion. \square

We easily find that (a) $f(x) > 0$ for $x > 0$, and thanks to Lemma 4.1, (b) $g(x) > 0$ and (c) $g'(x) < 0$ for $x > 0$. The function F_{μ_2} vanishes at $-\frac{i}{2}$ since it is a pole of G_{μ_2} , and from properties (a) and (c), F_{μ_2} is injective on $\gamma_{1/2}$. Property (b) implies that $F_{\mu_2}(\gamma_{1/2}) \subset \overline{\mathbb{C}^-}$. Thus conditions (B) and (C) are satisfied.

The following properties can be proved from (4.1):

- (i) $\operatorname{Re} G_{\mu_2}(x + yi) > 0$ for $x > 0$ and $y \geq -\frac{1}{2}$;
- (ii) $\operatorname{Im} G_{\mu_2}(yi) < 0$ for $y > -\frac{1}{2}$.

So G_{μ_2} does not have a zero in $\overline{D(\gamma_{1/2})}$ and so F_{μ_2} is analytic in $D(\gamma_{1/2})$, continuous on $\overline{D(\gamma_{1/2})}$. Consequently $\gamma_{1/2} = \{x - \frac{i}{2} : x \in \mathbb{R}\}$ satisfies condition (D).

Condition (E) is proved similarly to the case of ρ_t .

Open problems. The authors have not been able to solve the following questions.

- (a) Free infinite divisibility for Meixner distributions ρ_t in the case $t > \frac{1}{2}$ and for non symmetric Meixner distributions.
- (b) Free infinite divisibility for the measure with density $\frac{2\pi}{2^r B(\frac{r}{2}, \frac{r}{2})} \left(\frac{1}{\cosh \pi x} \right)^r$, $r > 0$.
- (c) Characterization of the class \mathcal{UI} in terms of free Lévy measures.
- (d) Combinatorial meaning of the free cumulant sequence of ρ_t , in particular of $\rho_{1/2}$.

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